$\boxed{\text{Day 8 - AM}}$

if A has an <u>inverse</u> A^{-1}, it is unique and

$$A^{-1}A = AA^{-1} = I$$

Yesterday, we saw for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, that A^{-1} exists if $\det A \neq 0$ and $A^{-1} = \dfrac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

An <u>elementary matrix</u> is one that is obtained by performing a single elementary row operation on an identity matrix.

We use elementary matrices to find inverses of larger $n \times n$ matrices. How?

Start with $[A \; I]$ and row reduce.

if A has an inverse, then $[A \; I] \sim [I \; A^{-1}]$

otherwise, A does not have an inverse.

<u>Ex</u>: Find the inverse of $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ if it exists

$$[A \; I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-4R_1 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \xrightarrow{3R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] \xrightarrow[-2R_3 + R_2]{-3R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right] = [I \; A^{-1}]$$

So $A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix}$

$\longrightarrow$

but the usual inverse only works for some $n \times n$ matrices.... is there a more general version?

Generalized Inverses

given A an $m \times n$ matrix, then an $n \times m$ matrix A^g is the generalized inverse if $AA^gA = A$

A^g is a generalized reflexive inverse if

1) $AA^gA = A$
2) $A^gAA^g = A^g$

optional conditions that may also occur

3) $(AA^g)^T = AA^g$
4) $(A^gA)^T = A^gA$ *symmetric!

a generalized inverse A^g satisfying all 4 of these properties is called the Moore-Penrose Pseudoinverse of A, denoted A^+

Thm: if A has real or complex entries, A^+ exists and is unique.

Ex: $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Verify $A^+ = \begin{bmatrix} 1/5 & 2/5 \end{bmatrix}$

$AA^+A = A$: $\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ✓

$A^+AA^+ = A^+$: $\begin{bmatrix} 1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\begin{bmatrix} 1/5 & 2/5 \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}\begin{bmatrix} 1/5 & 2/5 \end{bmatrix} = \begin{bmatrix} 1/5 & 2/5 \end{bmatrix}$ ✓

$(AA^+)^T = AA^+$: $(AA^+)^T = \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{bmatrix}^T = AA^+$ symmetric ✓

$(A^+A)^T = A^+A$: $(A^+A)^T = \begin{bmatrix} 1 \end{bmatrix}^T = \begin{bmatrix} 1 \end{bmatrix} = A^+A$ symmetric ✓